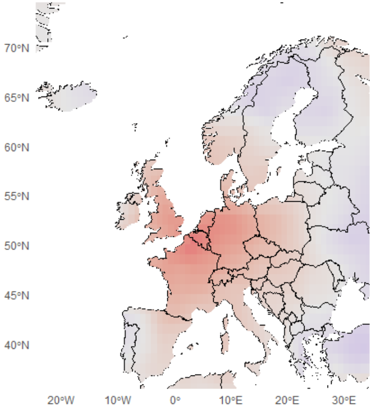
A spatio-temporal statistical framework for heatwave attribution under climate change



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EXALT
Extreme weather Attribution at mid–and high-Latitudes using advanced statistical Techniques

FÉDÉRATION WALLONIE-BRUXELLES

UCLouvain

- **Context.** More frequent and more intense heatwaves in recent decades (Auld et al., 2023; Christidis et al., 2015).

Introduction

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- **Challenges.** spacetime dependence/nonstationarity

- **Stage 1: Statistical modelling**

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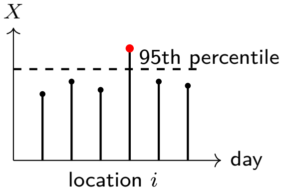
  - Simulate realistic heatwave events
  - Estimate return level curves for heatwave characteristics

Common ingredients of heatwaves (Perkins, 2015)

Most heatwave definitions combine three ingredients:

1. **High threshold**



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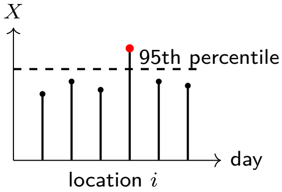

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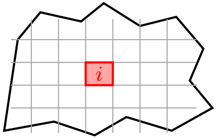

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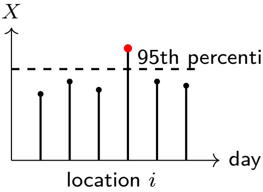

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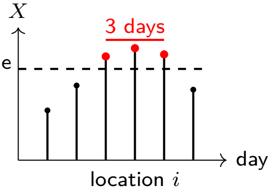
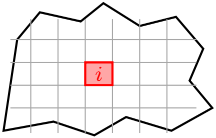
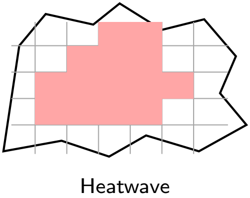




3. Spatial extent



Stage 1.1 Bayesian spatial quantile regression

How to model the change in exceedance probability?

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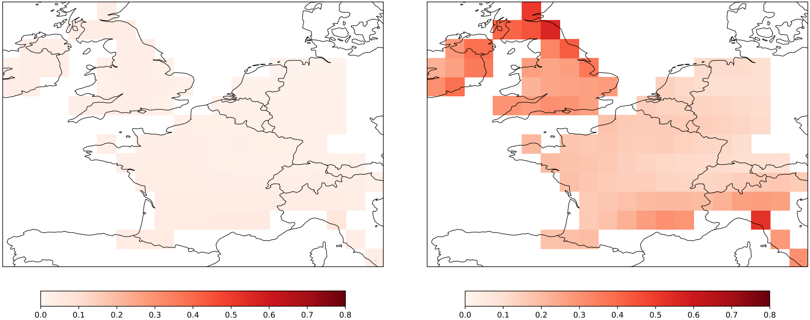
How to model the change in exceedance probability?
Conditional quantile model (Reich et al., 2011)

$$q(\tau \mid \mathrm{GMST}(t), \mathbf{s}) = \beta_0(\tau, \mathbf{s}) + \beta_1(\tau, \mathbf{s})\mathrm{GMST}(t)$$

Change in exceedance probabilities



No global warming

+3°C global warming

Change in the probability of exceeding a high threshold under different global warming scenarios.

Stage 1.2: Spatial GEV model (Cooley & Sain, 2010; Dyrrdal et al., 2015)

Marginal GEV model

$$M_{\mathbf{s},t} \sim \mathrm{GEV}\big(\mu_{\mathbf{s}}(t), \sigma_{\mathbf{s}}(t), \xi_{\mathbf{s}}\big)$$

$$\mu_{\mathbf{s}}(t) = \mu_{0,\mathbf{s}} + \mu_{1,\mathbf{s}}\, \mathrm{GMST}(t)$$

$$\log \sigma_{\mathbf{s}}(t) = \log \sigma_{0,\mathbf{s}} + \sigma_{1,\mathbf{s}}\, \mathrm{GMST}(t)$$

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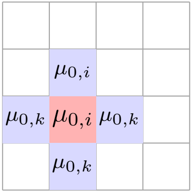
Spatial smoothing



Penalty

$$\sum_{k\sim i}(\mu_{0,i} - \mu_{0,k})^2$$

Stage 1.3: Space-time dependence model

Latent copula process (Dell'Oro & Gaetan, 2024; Huser & Wadsworth, 2019)

$$Z(\mathbf{s}, d) = R(\mathbf{s}, d)^{\delta}\, W(\mathbf{s}, d)^{1-\delta}, \qquad \delta \in [0, 1].$$

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- Copula of $Z(\mathbf{s}, d)$ models the dependence of $X(\mathbf{s}, d, t)$ within each summer season.

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Role of δ

$\delta > 0.5 \quad \Longrightarrow \quad$ Asymptotic Dependence (AD)

$\delta \leq 0.5 \quad \Longrightarrow \quad$ Asymptotic Independence (AI)

Stage 1.3: $Z(\mathrm{s}, d) = R(\mathbf{s}, d)^{\delta} W(\mathrm{s}, d)^{1-\delta}$

$$R(\mathbf{s}, d) = \frac{1}{1 - F_{R^*}(R^*(\mathbf{s}, d))}$$

$$R^*(\mathbf{s}, d) = \max_{k=1,\ldots,K} A_k \exp\left(-\frac{\|\boldsymbol{T}(\mathbf{s} - \boldsymbol{C}_k)\|}{\rho_s} - \frac{|d - T_k|}{\rho_t}\right)$$

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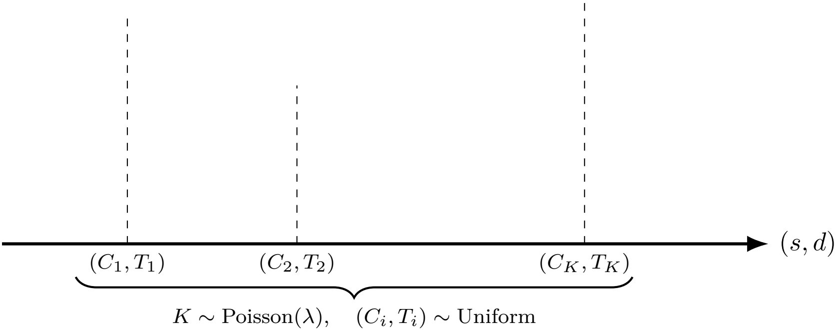

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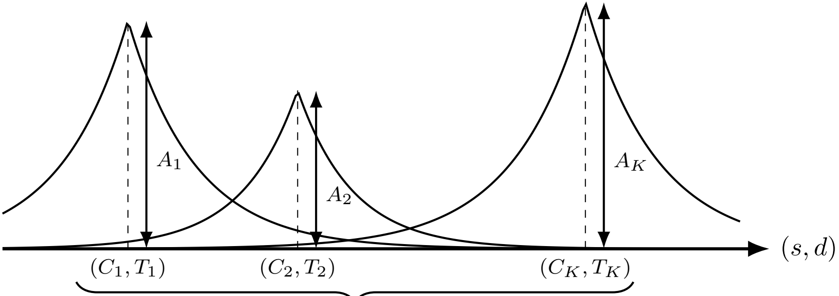
$$A_1, A_2, \ldots, A_K \overset{iid}{\sim} \text{Pareto}(1)$$



$$K \sim \text{Poisson}(\lambda), \quad (C_i, T_i) \sim \text{Uniform}$$

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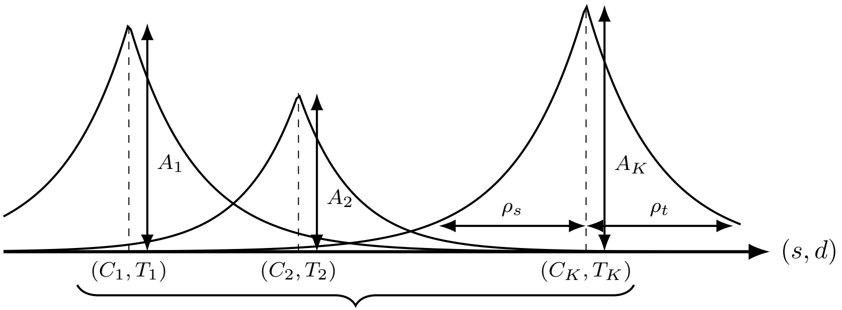

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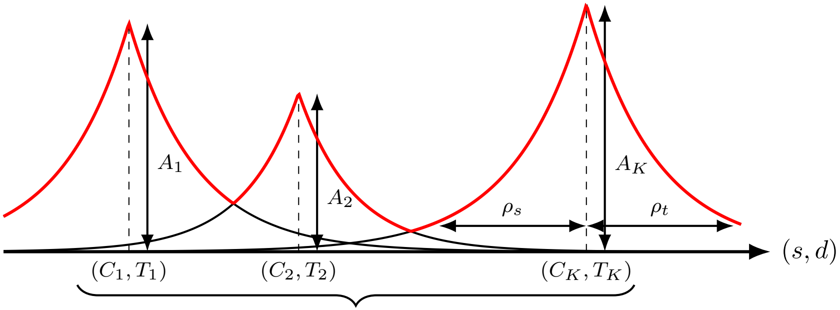

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Gaussian component

$$W(\mathbf{s}, d) = \frac{1}{1 - \Phi(W^*(\mathbf{s}, d))}, \qquad W^*(\mathbf{s}, d) \sim \mathcal{N}(0, \Sigma)$$

$$\Sigma = \Sigma_s \otimes \Sigma_d$$

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Space–time covariance structure

$$\Sigma_s(i, j) = \text{Cauchy}(\|\boldsymbol{T}(\mathbf{s}_i - \mathbf{s}_j)\|; \theta, \omega, \eta)$$

$$\Sigma_d(i, j) = \text{Matérn}(|d_i - d_j|; \kappa, \nu)$$

$\boldsymbol{T}$ rotates and rescales the spatial coordinate system.

Stage 2: Simulating temperature fields

1. **Simulate from the dependence model**

$$Z(\mathbf{s}, d, t) = R(\mathbf{s}, d, t)^{\delta} W(\mathbf{s}, d, t)^{1-\delta}, \qquad U(\mathbf{s}, d, t) = F_Z(Z(\mathbf{s}, d, t))$$

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2. Back-transform to the temperature scale

For each simulated value $U(\mathbf{s}, d, t)$:

$$X(\mathbf{s}, d, t) = \begin{cases} \text{BSQR model}, & U(\mathbf{s}, d, t) < \tau_{\mathbf{s},t}, \\ \text{EVT model}, & U(\mathbf{s}, d, t) \geq \tau_{\mathbf{s},t}. \end{cases}$$

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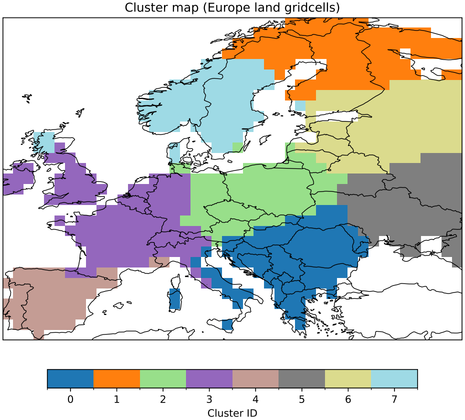
Bulk values are generated with BSQR; tail values are generated with EVT.

Data application

Application on MRI–ESM2: Cluster 3



Clustering strategy

- Extremal dependence between locations measured using the tail coefficient χ.
- Pairwise dissimilarities clustered using a k-*medoids* algorithm.

Following Kiriliouk and Naveau (2020), Bernard et al. (2013).

Estimated dependence parameters (MRI–ESM2)

Neural Bayes estimator used for the dependence parameters (Vihrs, 2022; Zammit-Mangion et al., 2025)

Parameter	Estimate	95% CI
δ [–]	0.45	(0.42, 0.47)
θ [km]	661.91	(638.46, 718.54)
κ [d]	21.00	(20.08, 23.88)
η [–]	2.04	(1.98, 2.23)
ω [rad]	2.33	(2.24, 2.37)
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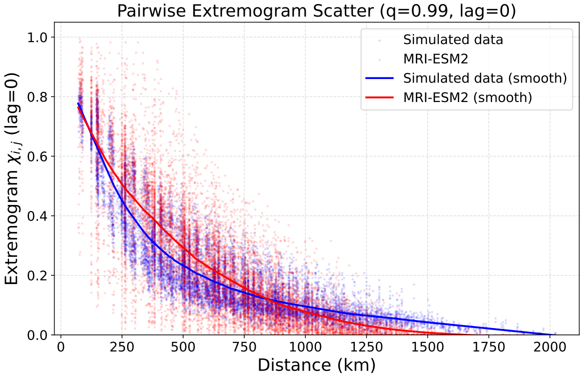
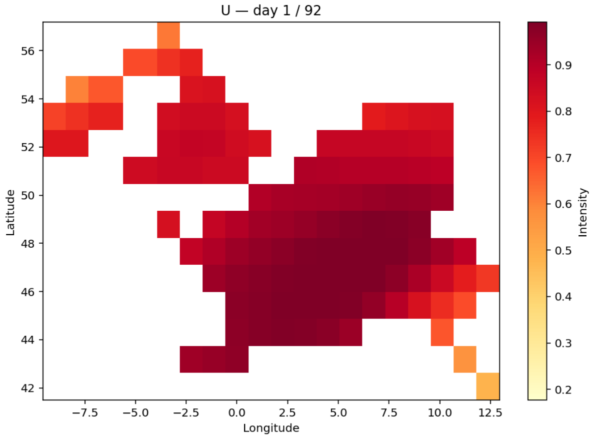
Extremal dependence in Cluster 3

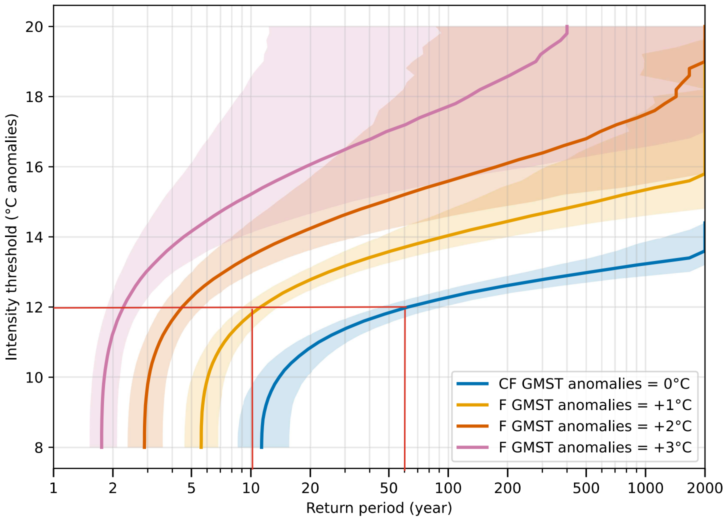


$$\chi(h) = \lim_{u \to 1} \mathsf{P}(F(X(\mathbf{s}+h)) > u \mid F(X(\mathbf{s})) > u)$$

Probability of observing a simultaneous extreme event at distance h.


U — day 1 / 92
56
54
52
50
48
46
44
42
Latitude
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
12.5
Longitude
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
Intensity

Heatwave: Return level (max intensity)



Conclusion

For details, see our paper on arXiv:
A spatio-temporal statistical framework for heatwave attribution under climate change.

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Factual vs. Counterfactual worlds

- **MRI-ESM2 (CMIP6) simulations**

$$Y^{\mathrm{F}}(\mathrm{s}, d, t) : \texttt{historical + SSP245}, \qquad Y^{\mathrm{C}}(\mathrm{s}, d, t) : \texttt{hist-nat}.$$

Factual world: natural + anthropogenic forcings; counterfactual world: natural forcings only.

Factual vs. Counterfactual worlds

- **MRI-ESM2 (CMIP6) simulations**

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  Factual world: natural + anthropogenic forcings; counterfactual world: natural forcings only.

- **Temperature anomalies**

  Daily temperatures are transformed into anomalies using the first $N = 100$ years (1850–1949):

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- **Purpose**

  Comparing factual and counterfactual anomalies quantifies the influence of anthropogenic forcing on heat extremes.

Common ingredients of heatwaves

Common ingredients in most heatwave definitions:

- High **threshold** relative to local climatology:

$$X(\mathbf{s}, d, t) \geq \vartheta_{\mathbf{s}}^{\text{thres}} = \inf\left\{x \in \mathbb{R} : \mathsf{P}\left(X^{\mathrm{C}}(\mathbf{s}, d, t) \leq x\right) \geq 0.95\right\}.$$

- **Duration**/persistence over consecutive days:

$$H(\mathbf{s}, d, t) = \mathbb{1}\left\{X(\mathbf{s}, d-2, t) \geq \vartheta_{\mathbf{s}}^{\text{thres}},\ X(\mathbf{s}, d-1, t) \geq \vartheta_{\mathbf{s}}^{\text{thres}},\ X(\mathbf{s}, d, t) \geq \vartheta_{\mathbf{s}}^{\text{thres}}\right\}$$

- **Spatial extent** of concurrent exceedances:

$$H_{\mathcal{R}}(d, t) = \mathbb{1}\left\{\sum_{\mathbf{s} \in \mathcal{R}} a(\mathbf{s}) H(\mathbf{s}, d, t) \geq \alpha \sum_{\mathbf{s} \in \mathcal{R}} a(\mathbf{s})\right\}$$

$\mathbf{s}$	spatial grid cell	$\mathbb{1}\{\cdot\}$	indicator function
d	summer day	$H(\mathbf{s}, d, t)$	local heatwave indicator
t	year	$\mathcal{R}$	spatial region
$X(\mathbf{s}, d, t)$	temperature anomaly	$a(\mathbf{s})$	grid-cell surface area
$\vartheta_{\mathbf{s}}^{\text{thres}}$	high-temperature threshold	α	minimum affected area fraction

Stage 1.1 Bayesian spatial quantile regression

Conditional quantile model (Reich et al., 2011)

$$q(\tau \mid \mathrm{GMST}(t), \mathbf{s}) = \beta_0(\tau, \mathbf{s}) + \beta_1(\tau, \mathbf{s})\mathrm{GMST}(t)$$

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Smooth quantile functions

Quantile coefficients are expanded using Bernstein polynomials:

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Key properties

- Non-crossing quantiles through increment parametrization
- Spatial smoothing via Gaussian process priors

Stage 1.2: Spatial GEV model (Cooley & Sain, 2010; Dyrrdal et al., 2015)

1. Marginal GEV model

$$M_{\mathbf{s},t} \sim \mathrm{GEV}\big(\mu_{\mathbf{s}}(t), \sigma_{\mathbf{s}}(t), \xi_{\mathbf{s}}\big)$$

$$\mu_{\mathbf{s}}(t) = \mu_{0,\mathbf{s}} + \mu_{1,\mathbf{s}}\,\mathrm{GMST}(t)$$

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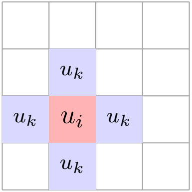
2. Spatial smoothing

$$\gamma_{\ell,\mathbf{s}} = \gamma_{\ell,g} + s_\ell u_{\ell,\mathbf{s}}, \qquad \boldsymbol{u}_\ell \sim \mathrm{ICAR}(\mathcal{A})$$



Penalty

$$\sum_{k \sim i} (u_i - u_k)^2$$

Stage 2: Simulating Temperature Field

1. **Simulate from the dependence model**

$$Z(\mathbf{s},d,t) = R(\mathbf{s},d,t)^{\delta} W(\mathbf{s},d,t)^{1-\delta}, \qquad U(\mathbf{s},d,t) = F_Z(Z(\mathbf{s},d,t))\,.$$

2. **Back-transform to the temperature scale**

$$X(\mathbf{s},d,t) = \begin{cases} q(U(\mathbf{s},d,t) \mid \mathbf{c}(t),\mathbf{s})\,, & U(\mathbf{s},d,t) < \tau_{\mathbf{s},t}, \\ \vartheta_{\mathbf{s}}^{\text{thres}} + \frac{\tilde{\sigma}_{\mathbf{s},t}}{\xi_{\mathbf{s}}}\left[\left(\frac{1-U(\mathbf{s},d,t)}{1-\tau_{\mathbf{s},t}}\right)^{-\xi_{\mathbf{s}}} - 1\right], & U(\mathbf{s},d,t) \geq \tau_{\mathbf{s},t},\ \xi_{\mathbf{s}} \neq 0. \end{cases}$$

F_Z: marginal distribution of Z; q: fitted BSQR quantile function;

$\tau_{\mathbf{s},t} = q^{-1}(\vartheta_{\mathbf{s}}^{\text{thres}} \mid \text{GMST(t)}, \mathbf{s})$

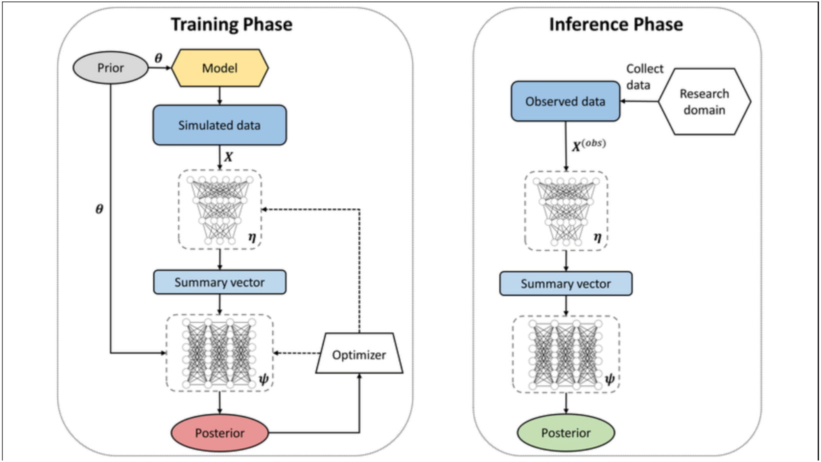
$\tilde{\sigma}_{\mathbf{s},t} = \sigma_{\mathbf{s}}(t) + \xi_{\mathbf{s}}\{\vartheta_{\mathbf{s}}^{\text{thres}} - \mu_{\mathbf{s}}(t)\}.$

Neural Bayes estimator

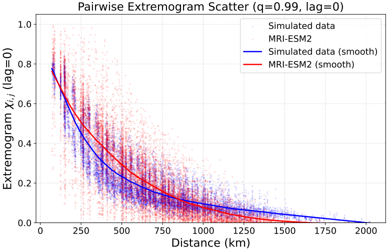
Estimation: likelihood-free approach.

- **Neural Bayes estimator** (Sainsbury-Dale et al., 2024): simulate from the model under priors on $\boldsymbol{\Delta} = (\delta, \omega, \rho, \theta, \lambda, \kappa)^\top$; train a network to predict $\boldsymbol{\Delta}$ from summary statistics.
- **Summary statistics used (space, time, spacetime)**
  - χ-grids at $u \in \{0.90, 0.95, 0.99\}$ over distance bins (100 km) and temporal lags $h = 0{:}10$.
  - Auto-tail dependence function (ATDF) vs. lag h.
  - Spatial extremogram quantile profile (10/50/90% by distance bin).
  - Variogram map of rank-uniform field to diagnose anisotropy.
- **Benefit.** Avoids costly likelihood inference, scales to high dimension.

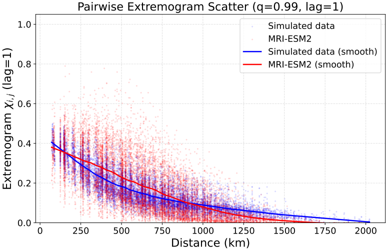
Amortized inference: overview figure



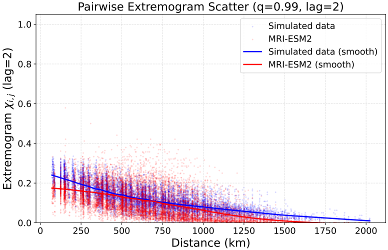
Source: Radev et al., Amortized Bayesian Inference for Models of Cognition.



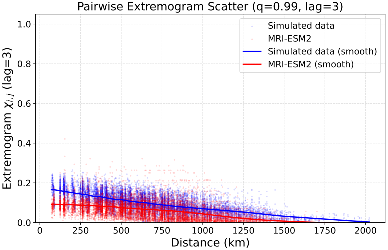
(a) Lag 0



(b) Lag 1



(c) Lag 2



(d) Lag 3

Stage 2: Return periods

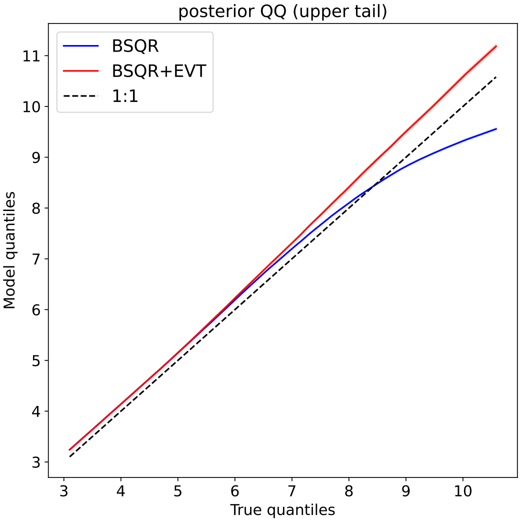
Return periods and levels.

- The *return level* z_T is the value exceeded on average once every T years:

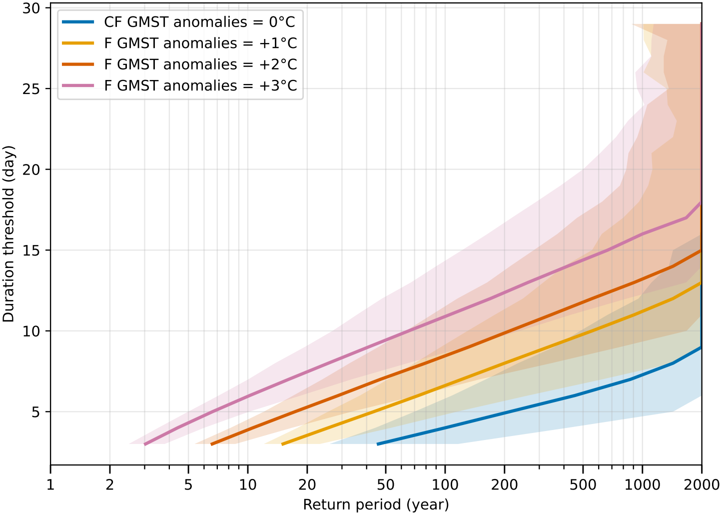
$$\mathsf{P}(X > z_T) = \frac{1}{T}.$$

- Example: If the 100-year return level of annual maximum temperature is $z_{100} = 40^\circ$C, this means
  - On average, a temperature higher than 40°C will be observed once every 100 years.
  - Equivalently, in any given year, the probability of exceeding 40°C is $1/100 = 0.01$.

Why BSQR + EVT


posterior QQ (upper tail)
BSQR
BSQR+EVT
1:1
Model quantiles
True quantiles
3
4
5
6
7
8
9
10
11

Heatwave: Duration



EXALT
Extreme weather Attribution at mid-and high-Latitudes using advanced statistical Techniques

FÉDÉRATION WALLONIE-BRUXELLES

UCLouvain